FILL RADIUS AND THE FUNDAMENTAL GROUP

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Abstract. In this note we relate the geometric notion of fill radius with the fundamental group of the manifold. We prove: Suppose that a closed Riemannian manifold M satisfies the property that its universal cover has bounded fill radius. Then the fundamental group of M is virtually free. We explain the relevance of this theorem to some conjectures on positive isotropic curvature and 2-positive Ricci curvature.

1. **Introduction**

Let (M,g) be an n-dimensional Riemannian manifold. The notion of *fill radius*, introduced in [G1], [G-L], [S-Y], is a type of "two-dimensional diameter". Let γ be a smooth simple closed curve in M which bounds a disk in M. Set $N_r(\gamma) = \{x \in M : d(x,\gamma) \leq r\}$. We define the *fill radius of* γ to be:

$$\text{fillrad}(\gamma) = \sup\{r : \text{dist}(\gamma, \partial M) > r \text{ and } \gamma \text{ does not bound a disc in } N_r(\gamma)\}$$

We say a Riemannian manifold (M,g) has its *fill radius bounded by* C if every smooth simple closed curve γ which bounds a disk in M satisfies,

$$\text{fillrad}(\gamma) \leq C.$$

Clearly if the diameter of (M,g) is bounded so is its fill radius. In particular if for all $p \in M$, $\text{Ric}(p) \geq \alpha$, where α is a positive constant, then there is a constant $C = C(\alpha)$ such that the fill radius of M is bounded by C. It is an interesting problem to find "positive curvature conditions" that imply fill radius bounds. In [G-L] and [S-Y] versions of the following result on positive scalar curvature and fill radius are proved. (Throughout this introduction, for technical reasons related to the solution of the Plateau problem, if (M,g) is not compact we will assume it is complete, its sectional curvature is bounded above and its injectivity radius is bounded below away from zero, i.e., we will assume that (M,g) has bounded geometry. If (M,g) is a cover of a closed Riemannian manifold then these conditions are satisfied.)

Theorem 1.1 (Gromov-Lawson, Schoen-Yau)**.** *Let (M,g) be a complete Riemannian three manifold with positive scalar curvature S that satisfies $S \geq \alpha$, for a constant $\alpha > 0$. Then if γ is a smooth simple closed curve in M which bounds a disk in M:*

$$\text{fillrad}(\gamma) \leq \sqrt{\frac{8}{3}}\frac{\pi}{\sqrt{\alpha}}$$

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We next recall two positive curvature conditions that conjecturally imply fill radius bounds. We say (M, g) has *two-positive Ricci curvature* if at each point $p \in M$ the sum of the two smallest eigenvalues of the Ricci curvature at p is positive. We say that the two-positive Ricci curvature is bounded below by α if the sum of the two smallest eigenvalues is greater than α. It has been conjectured by the second author [W] that:

Conjecture 1.1. *Let (M, g) be a complete Riemannian n-manifold with two-positive Ricci curvature is bounded below by α, for a constant $\alpha > 0$. Then if γ is a smooth simple closed curve in M which bounds a disk in M:*

$$\mathrm{fillrad}(\gamma) \leq C(\alpha)$$

We say (M, g) has *positive isotropic curvature bounded below by* α if at each point $p \in M$ and for every orthonormal four frame $\{e_1, e_2, e_3, e_4\}$ the curvature satisfies:

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} + 2R_{1234} \geq \alpha.$$

It has been conjectured (at least, implicitly by Gromov [G2], Fraser [F]) that:

Conjecture 1.2. *Let (M, g) be a complete Riemannian n-dimensional manifold with positive isotropic curvature bounded below by α, for a constant $\alpha > 0$. Then if γ is a smooth simple closed curve in M which bounds a disk in M:*

$$\mathrm{fillrad}(\gamma) \leq C(\alpha)$$

A group G is said to be *virtually free* if it possesses a finite index subgroup that is a free group. If G is the fundamental group of a manifold M then G is virtually free if some finite cover of M has fundamental group that is a free group.

In this note we prove:

Theorem 1.2. *Let M be a closed Riemannian n-manifold. Suppose that the universal cover $\pi : \tilde{M} \to M$ is given the Riemannian metric $\tilde{g}$ such that π is a local isometry. If $(\tilde{M}, \tilde{g})$ has bounded fill radius then the fundamental group of M is virtually free.*

If Conjecture 1.1 is true then Theorem 1.2 implies that the fundamental group of a closed n-manifold with two-positive Ricci curvature is virtually free. If Conjecture 1.2 is true then Theorem 1.2 implies that the fundamental group of a closed n-manifold with positive isotropic curvature is virtually free. We remark that, based on the work of Micallef-Wang [M-W], Gromov [G2] and Fraser [F] explicitly conjecture that the fundamental group of a closed n-manifold with positive isotropic curvature is virtually free. However, in light of Theorem 1.2, we attribute Conjecture 1.2 as above.

We are indebted to Bruce Kleiner for pointing out that a homological version of Theorem 1.2 can be proved using our techniques. This is outlined in Section 3. The second author wishes to thank Nick Ivanov for useful discussions.

2. **Fill Radius and the Fundamental Group**

In this section we give the proof of Theorem 1.2. Our approach is based on the notion of the number of ends of a group G. There are various definitions of this notion. For our purposes the following definition will suffice:

Definition 2.1. Given a group G we define the number of ends, $e(G)$, of G to be the number of topological ends of $\tilde{K}$, where $\tilde{K} \to K$ is a regular covering of the finite simplicial complex K by the simplicial complex $\tilde{K}$ and G is the group of covering transformations.

In particular, if G is the fundamental group of a closed manifold N then the number of ends of G is the number of ends of the universal cover $\tilde{N}$ of N. It is not difficult to show that a group G can have 0,1,2 or infinitely many ends [E].

We will need the following three lemmas.

Lemma 2.1. *Let N be a closed manifold. Suppose that $N_0 \to N$ is a covering of N such that N_0 has fundamental group G that is finitely generated and has exactly one end. Let γ be a simple closed curve in N_0 that represents an infinite order generator $[\gamma]$ of G. Let $\tilde{N} \to N_0$ be the universal cover and let $\tilde{\gamma}$ be the lift of γ to $\tilde{N}$. Then the two ends of $\tilde{\gamma}$ lie in the same end of $\tilde{N}$.*

Proof. There is a finite simplicial complex K with regular covering $\tilde{K}$ such that G acts as the group of covering transformations. There is an imbedding $\imath : K \to N_0$ that induces an epimorphism of fundamental groups. In particular, the generators of G all lie in K. Then there is an imbedding $\tilde{\imath} : \tilde{K} \to \tilde{N}$. If $B \subset \tilde{N}$ is compact then $\tilde{\imath}^{-1}(B) \subset \tilde{K}$ is compact.

Let γ be a simple closed curve in N_0 that represents an infinite order generator $[\gamma]$ of G. After a homotopy the lift $\tilde{\gamma}$ can be assumed to lie in $\tilde{K}$. Since G has exactly one end, any two points on $\tilde{\gamma}$, not in $\tilde{\imath}^{-1}(B)$, can be joined by a curve α in $\tilde{K} \setminus \tilde{\imath}^{-1}(B)$. The curve $\tilde{\imath}(\alpha)$ then lies in $\tilde{N} \setminus B$ and joins points on $\tilde{\gamma}$ not in B. Since this is true for any compact set B the conclusion follows. $\square$

The next lemma is a version of Lemma 2.1 for torsion elements that are sufficiently long.

Lemma 2.2. *Let N be a closed Riemannian manifold. Suppose that $N_0 \to N$ is a covering of N such that N_0 has fundamental group G that is a finitely generated infinite group with exactly one end. Let $g_i \in G$ be a sequence and suppose that each g_i is represented by a closed curve γ_{g_i} beginning and ending at $q \in N_0$ such that each curve lies in a fixed compact region S. Let $\tilde{N} \to N_0$ be the universal cover and let $\tilde{\gamma}_{g_i}$ be a lift of γ_{g_i} to $\tilde{N}$. Denote the distance between the endpoints of $\tilde{\gamma}_{g_i}$ by d_i and suppose that $d_i \to \infty$. Let x_i be a point on $\tilde{\gamma}_{g_i}$ such that the distance between x_i and each endpoint is at least $\frac{d_i}{2}$. In addition, suppose that all the x_i lie in a small coordinate ball $B_\delta(x)$. Then given $B_R(x)$, $R >> \delta$, for sufficiently large i the endpoints of $\tilde{\gamma}_{g_i}$ can be joined by a path in $\tilde{N} \setminus B_R(x)$.*

Proof. For sufficiently large i neither endpoint of $\tilde{\gamma}_{g_i}$ lies in a relatively compact region of $\tilde{N} \setminus B_R(x)$. Using the same notation as in Lemma 2.1, after a homotopy the curve $\tilde{\gamma}_{g_i}$ can be assumed to lie in $\tilde{K}$. Since G has exactly one end the endpoints of $\tilde{\gamma}_{g_i}$ lie in the same end of $\tilde{K}$ and therefore can be joined by a curve α in $\tilde{K} \setminus$

$\tilde{\imath}^{-1}(B_R(x))$. The curve $\tilde{\imath}(\alpha)$ then lies in $\tilde{N} \setminus B_R(x)$ and joins the endpoints of $\tilde{\gamma}_{g_i}$. □

Lemma 2.3. *Let M be a complete manifold with finitely generated fundamental group G. Then there is a compact subset S of M such that every element of G can be represented by a closed curve beginning and ending at $q \in S$ that lies entirely in S.*

Proof. Choose a finite set of generators and represent each generator by a smooth closed curve beginning and ending at $q \in M$. Then each curve lies in a fixed compact set S. The result follows. □

Theorem 2.4. *Let N be a closed Riemannian manifold. Suppose that the universal cover $\tilde{N}$ has the property that the fill radius of every simple closed curve is uniformly bounded above. If G is a finitely generated subgroup of $\pi_1(N)$ then G cannot have exactly one end.*

Proof. Assume, by way of contradiction, that the subgroup G of $\pi_1(N)$ has exactly one end. Let M be a covering of N with fundamental group $\pi_1(M)$ isomorphic to G. If G contains an element of infinite order the proof is simpler. We begin with this case though, strictly speaking, this is not necessary.

Assume that G contains a generator of infinite order and denote by γ a minimal geodesic in M that represents this generator. Let $p : \tilde{N} \to M$ be the universal cover and let $\tilde{\gamma}$ be the geodesic line that is a lift to $\tilde{N}$ of γ. Let $x \in \tilde{\gamma}$ and $B_R(x) \subset \tilde{N}$ be the metric ball of radius R, center x. Then because G has exactly one end by Lemma 2.1 both ends of $\tilde{\gamma}$ in $\tilde{N} \setminus B_R(x)$ lie in the same end of $\tilde{N}$. The geodesic line $\tilde{\gamma}$ consist of two geodesic rays $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ beginning at x. For $j = 1, 2$, choose a point $p_j \in \tilde{N} \setminus B_R(x)$ along $\tilde{\gamma}_j$ and denote the segment of $\tilde{\gamma}_j$ from x to p_j by τ_j. Since p_1 and p_2 lie in the same end there is a curve $\beta \subset \tilde{N} \setminus B_R(x)$ joining p_1 and p_2. Denote the closed curve $\tau_1 \cup \beta \cup \tau_2$ by η. Since $\tilde{N}$ is simply connected η is null homotopic and has fill radius greater than $\frac{R}{2}$. For sufficiently large R this contradicts the fill radius bound.

Next assume that G has no elements of infinite order but that G is infinite. Since G is finitely generated there is a point $q \in M$ and a ball $B_r(q) \subset M$ such that every element $g \in G$ can be represented by a closed curve γ_g in $B_r(q)$ beginning and ending at q. Let $\tilde{q}$ denote a lift of q and denote by $\tilde{\gamma}_g$ the lifts of the γ_g to $\tilde{N}$ that begin at $\tilde{q}$. The endpoints of $\tilde{\gamma}_g$ are the points $\tilde{q}$ and $g(\tilde{q})$. Since G is infinite we can choose a sequence $g_i \in G$ such that the distance $d(\tilde{q}, g_i(\tilde{q})) = d_i \to \infty$. Choose a point x_i on $\tilde{\gamma}_{g_i}$ such that $d(x_i, \tilde{q}) \geq \frac{d_i}{2}$ and $d(x_i, g_i(\tilde{q})) \geq \frac{d_i}{2}$. Using the Deck transformations find elements h_i of G that move the points x_i into a fixed fundamental region U containing $\tilde{q}$. Denote the curves $h_i(\tilde{\gamma}_{g_i})$ by σ_i. Then the endpoints of σ_i remain at least $\frac{d_i}{2}$ distant from $h_i(x_i)$ and are distance d_i from each other. Note that the sequence $\{h_i(x_i)\}$ lies in the compact set $\bar{U} \cap p^{-1}(B_r(q))$. Choosing a subsequence of $\{h_i(x_i)\}$ we can suppose that the sequence $\{h_i(x_i)\}$ converges to $x \in \tilde{N}$ and therefore that $h_i(x_i) \in B_\delta(x)$, for some $\delta > 0$. Consider the ball $B_R(x) \in \tilde{N}$, where $R >> \delta$. Denote the endpoints of σ_i by y_i and z_i. For i sufficiently large, y_i and z_i lie outside $B_R(x)$ and do not lie in any relatively

compact region of $\tilde{N} \setminus B_R(x)$. Thus, by Lemma 2.2, y_i and z_i can be joined by a smooth curve α_i lying in $\tilde{N} \setminus B_R(x)$. Join x to y_i by a minimal geodesic τ_i and join x to z_i by a minimal geodesic ρ_i. The closed loop $\tau_i \cup \alpha_i \cup \rho_i$ is null homotopic and has fill radius greater than $\frac{R}{2}$. For R sufficiently large this contradicts the fill radius bound.

Finally if G is finite then G has zero ends. □

To prove our next result we will use work of Dunwoody [D]. Stallings' Structure theorem [St1] for finitely generated groups with more than one end is formulated in [D] as follows: *Let G be a finitely generated group. Then $e(G) > 1$ if and only if there is a G-tree T such that the stabilizer G_e of each edge e is finite and the stabilizer G_v of each vertex v is finitely generated and $G_v \neq G$.*

Definition 2.2. A finitely generated group G is said to be *accessible* if there is a G-tree T such that G_e is finite for each edge e of T and G_v has at most one end for each vertex v of T.

Dunwoody's main result in [D] is: *A finitely presented group G is accessible.* (also, see [D-D] Chap. 6 Theorem 6.3).

Theorem 2.5. *Let N be a closed Riemannian manifold. Suppose that the universal cover $\pi : \tilde{N} \to N$ is given the Riemannian metric $\tilde{g}$ such that π is a local isometry. If $(\tilde{N}, \tilde{g})$ has fill radius bounded above then the fundamental group $\pi_1(N)$ is virtually free.*

Proof. By Theorem 2.4, $G = \pi_1(N)$ has no finitely generated subgroups with exactly one end. Since $\pi_1(N)$ is finitely presented, by Dunwoody's result, it is accessible. Therefore there is a G-tree T such that G_e is finite for each edge e of T and G_v is finite for each vertex v of T. Then, by [Se] (see Chap. II, Sec. 2.6, Prop. 11), it follows that G is virtually free. □

Under a more restrictive condition on the fundamental group a better result is available.

Theorem 2.6. *Let N be a closed Riemannian manifold with torsion-free fundamental group. Suppose that the universal cover $\pi : \tilde{N} \to N$ is given the Riemannian metric $\tilde{g}$ such that π is a local isometry. If $(\tilde{N}, \tilde{g})$ has fill radius bounded above then the fundamental group $\pi_1(N)$ is free of finite rank.*

Proof. We use Grushko's Theorem (see [Ma]) and the following theorem of Stallings [St2] (also, [D-D] Chap. 4 Theorem 6.10): *If G is a torsion-free, finitely generated group with infinitely many ends then G is a non-trivial free product.* Applying Stallings' theorem to $G = \pi_1(N)$, we have $G \simeq G_1 * G_2$, where each G_i is finitely generated (by Grushko's Theorem) and each G_i has either two or infinitely many ends (by Theorem 2.4). Then apply Stallings theorem to each G_i with infinitely many ends and iterate. By Grushko's Theorem, this process terminates after finitely many steps resulting in $G \simeq G_1 * \cdots * G_k$, where each G_i is finitely generated and has two ends. Since a torsion-free, finitely generated group with two ends is infinite cyclic, we conclude that $G = \pi_1(N)$ is a free group of finite rank. □

3. Homology and Fill Radius

In this section we describe the analog of the previous results for the notion of *homological fill radius*. We will continue to work with n-dimensional Riemannian manifolds (M, g) though the results we describe can be formulated for more general spaces. Let Γ be a one-cycle which bounds in M. Set $N_r(\Gamma) = \{x \in M : d(x, \Gamma) \leq r\}$. We define the *homological fill radius of* Γ to be:

$$\mathrm{H}_1\mathrm{fillrad}(\Gamma) = \sup\{r : \mathrm{dist}(\Gamma, \partial M) > r \text{ and } \Gamma \text{ does not bound in } N_r(\Gamma)\}$$

We say a Riemannian manifold (M, g) has its *homological fill radius bounded by* C if every one-cycle Γ which bounds in M satisfies,

$$\mathrm{H}_1\mathrm{fillrad}(\Gamma) \leq C.$$

Our main theorem is:

Theorem 3.1. *Let (X, g) be a complete Riemannian n-manifold with bounded geometry, with $H_1(X, \mathbb{Z}) = 0$ and that satisfies a homological fill radius bound. Suppose the group G acts freely, properly discontinuously and co-compactly on X. Then G is virtually free. If, in addition, G is torsion free then G is free of finite rank.*

The first step in the proof of the theorem is the analog of Theorem 2.4.

Theorem 3.2. *Let (X, g) be a complete Riemannian n-manifold with bounded geometry, with $H_1(X, \mathbb{Z}) = 0$ and that satisfies a homological fill radius bound. Suppose the group G acts freely, properly discontinuously and co-compactly on X. If H is a finitely generated, infinite subgroup of G then H cannot have exactly one end.*

Proof. To begin we observe, without loss of generality, that we can suppose that G acts as a group of isometries. To see this, note that given a metric on X/G it lifts to a complete metric with bounded geometry $\bar{g}$ on X. By assumption (X, g) is a complete Riemannian manifold with bounded geometry. Therefore $\bar{g}$ and g are quasi-isometric. The homological fill radius bound is a quasi-isometry invariant so $(X, \bar{g})$ satisfies this condition.

Suppose, by way of contradiction, that H has exactly one end. The quotient space $N = X/G$ is a compact Riemannian manifold with G a subgroup of the fundamental group. In particular, H is a finitely generated, infinite subgroup of the fundamental group. As in the proof of Theorem 2.4, let M be a covering of N with fundamental group $\pi_1(M)$ isomorphic to H. Note that since H is a subgroup of G, X is a regular covering space of M. Since H is finitely generated there is a point $q \in M$ and a ball $B_r(q) \subset M$ such that every element $h \in H$ can be represented by a closed curve γ_h in $B_r(q)$ beginning and ending at q. Let $\tilde{q} \in X$ denote a lift of q and denote by $\tilde{\gamma}_h$ the lifts of the γ_h to X that begin at $\tilde{q}$. Since H is infinite we can choose a sequence $h_i \in H$ such that the distance $d(\tilde{q}, h_i(\tilde{q})) = d_i \to \infty$. Choose a point x_i on $\tilde{\gamma}_{g_i}$ such that $d(x_i, \tilde{q}) \geq \frac{d_i}{2}$ and $d(x_i, h_i(\tilde{q})) \geq \frac{d_i}{2}$. On X the group H acts as Deck transformations so using the same argument as in the proof of Theorem 2.4 we can suppose that the points x_i lie in a ball $B_\delta(x)$ for some $x \in X$. Hence we find a sequence of curves σ_i in X with endpoints y_i and z_i and containing points x_i with the following properties: (i) all the x_i lie in a fixed coordinate ball $B_\delta(x)$ for some $x \in X$, (ii) $d(y_i, z_i) = d_i \to \infty$, as $i \to \infty$, (iii) $d(x_i, y_i) \geq \frac{d_i}{2}$ and $d(x_i, z_i) \geq \frac{d_i}{2}$. Given a metric ball $B_R(x) \subset X$ choose i sufficiently large such

that y_i and z_i lie outside $B_R(x)$ and do not lie in any relatively compact region of $X \setminus B_R(x)$.

To use the assumption that H has one end we let K be a finite simplicial complex with regular covering $\tilde{K}$ such that H acts as the group of covering transformations. There is an imbedding $\imath : K \to M$ that induces an epimorphism of fundamental groups. In particular, the generators of H all lie in K. Since H is a subgroup of G, there is an imbedding $\tilde{\imath} : \tilde{K} \to X$. Then, up to homotopy, σ_i is a path in $\tilde{\imath}(\tilde{K})$. For i sufficiently large the endpoints of $\tilde{\imath}^{-1}(\sigma_i)$ lie outside $\tilde{\imath}^{-1}(B_R(x))$ and do not lie in any relatively compact region of $\tilde{K} \setminus \tilde{\imath}^{-1}(B_R(x))$. Since H has exactly one end it follows that the endpoints of $\tilde{\imath}^{-1}(\sigma_i)$ can be joined by a curve lying in $\tilde{K} \setminus \tilde{\imath}^{-1}(B_R(x))$. Hence the endpoints of σ_i, y_i and z_i, can be joined by a smooth curve α_i lying in in $X \setminus B_R(x)$. Join x to y_i by a minimal geodesic τ_i and join x to z_i by a minimal geodesic ρ_i. The closed loop $\Gamma = \tau_i \cup \alpha_i \cup \rho_i$ is a one-cycle in X and since $H_1(X, \mathbb{Z}) = 0$, Γ spans a chain. The homological fill radius of Γ is greater than $\frac{R}{2}$. For R sufficiently large this contradicts the homological fill radius bound of X. □

Proof of Theorem 3.1. The group G need not be finitely presented however Dunwoody's work applies more generally to show that if G is almost finitely presented then G is accessible [D]. A group that acts freely, properly discontinuously and co-compactly on a space X with $H_1(X, \mathbb{Z}_2) = 0$ is almost finitely presented. Therefore G is accessible. By Theorem 3.2 no finitely generated subgroup of G has exactly one end. Hence there is a G-tree T such that G_e is finite for each edge e of T and G_v is finite for each vertex v of T. Then, by [Se], it follows that G is virtually free.

If G is torsion free the proof is identical to the proof of Theorem 2.6.

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